"Computation and Use of Expected Mean Squares in Analysis of Variance."

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Computation and Use of Expected Mean Squares in Analysis of Variance

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The available procedures for the computation and use of expected mean squares (EMS) in analysis of variance are reviewed. The discussion is tutorial in nature and includes algorithms for EMS computation, the selection of appropriate error terms in constructing F tests for the significance of factor effects, the estimation of variance components, synthesized mean squares and associated degrees of freedom and approximate F tests. Emphasis is on practical applications. Examples of both balanced and unbalanced designs are presented.

Introduction

ANALYSIS of variance (ANOVA) is a method of data analysis which is commonly used to test the significance of factor effects and to estimate variance components, i.e., the amount of the variability in the observations which can be attributed to the various sources of variability being studied. Variance components are particularly useful in designing sampling plans to monitor various product properties and in establishing quality control procedures.

When an ANOVA table has been constructed, one must refer to the expected values of the mean squares (EMS) in order to determine

(i) which mean squares should be compared to test the significance of the factor effects, and

(ii) which linear combinations of the observed mean squares should be formed to estimate the variance components.

The objective of this paper is to present a tutorial discussion of the computation and use of expected mean squares in ANOVA. Emphasis is on practical applications. A review of the available procedures and illustrative examples are included.

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KEY WORDS: Expected Mean Squares, Analysis of Variance, Variance Components, F Tests.

Expected Mean Square Computation

The data in Table 1 are the results of a 4×3 cross classification design with duplicate observations obtained at each of the $4 \cdot 3 = 12$ treatment combinations for a total of 24 observations. The ANOVA for these data is given below.

ANOVA TABLE

Source	df	SS	MS	F
Total	23	300		
A	3	120	40	
B	2	48	24	
AB	6	84	14	
Duplicates	12	48	4	

After an ANOVA table has been constructed, the EMS are examined to determine which F tests should be performed to test hypotheses concerning the sources of variation in the ANOVA table (i.e., the EMS indicate the appropriate error term for each source of variation). If A is a source of interest, then the numerator of the F ratio is the A mean square (MS_A) and the denominator of the F ratio is a mean square that has the same expected value as MS_A when A is assumed to have no effect ($H_0 : \sigma_A^2 = 0$).

An EMS is of the form

$$E(MS) = k_1\sigma_1^2 + k_2\sigma_2^2 + \cdots + k_p\sigma_p^2$$

TABLE 1. Two Factor Crossed Design

PRODUCTION RATES (CODED) FOR AN EXPERIMENT IN A CATALYST PLANT *

FACTOR A REAGENT	FACTOR B CATALYST C_1	C_2	C_3
R_1	4 6	11 7	5 9
R_2	6 4	13 15	9 7
R_3	13 15	15 9	13 13
R_4	12 12	12 14	7 9

* See Reference (23)

where E is the expected value operator, MS is the observed mean square, the k_i's are coefficients to be determined and the σ_i^2's are variance components of the various sources of variation being studied. If $k_i = 0$, then σ_i^2 is not included in the EMS. The k_i's are functions of

(i) the experimental design,
(ii) size of the various populations being sampled,
(iii) size of the sample drawn from the populations, and
(iv) the sampling pattern—balanced or unbalanced.

An important consideration is whether the design is balanced or unbalanced. The computation of the EMS for the balanced designs (crossed, nested, partially crossed and nested) is described in the classic paper by Cornfield and Tukey [3]. These procedures are also given in statistics texts by Bennett and Franklin [26], Hicks [28], and Scheffé [30]. Algorithms for the computation of EMS for unbalanced nested designs have been presented by Anderson and Bancroft [25], Mahamunulu [17], and Gaylor and Hartwell [9]. Anderson and Bancroft's algorithm, which was originally published by Ganguli [7], and Mahamunulu's procedure are applicable for infinite populations only. Gaylor and Hartwell's algorithm is applicable to both finite and infinite populations. It should be noted that, while Mahamunulu discusses the three-way nested design, his results can be generalized to nested designs with any number of factors. Gaylor, Lucas, and Anderson [11] have discussed the computation of the EMS for unbalanced crossed designs. Hartley [13] and Rao [18] have also discussed the computation of EMS for unbalanced designs. From this review of the literature, it is apparent that there are several algorithms available for the computation of EMS. It is this author's opinion that the following algorithms are most useful.

Design	*Algorithm*
Balanced	Cornfield & Tukey [3, 26, 28, 30]
Unbalanced Nested	Gaylor & Hartwell [9]
Other Unbalanced Designs	Gaylor, Lucas, & Anderson [11]

It should be noted that in the case of infinite populations, Gaylor and Hartwell's algorithm reduces to the procedure described by Ganguli [7], Anderson and Bancroft [25], and Mahamunulu [17]. In the following sections, these procedures will be illustrated in the computation of EMS for the balanced crossed design discussed earlier and an unbalanced nested design. The reader is referred to the paper by Gaylor, Lucas, and Anderson for an example of an unbalanced crossed design [11] References [6, 14, 21, 22, 27, 29, 32, 33] also discuss various aspects of the computation and use of expected mean squares.

Cross Classification Design

The computation of EMS for the balanced cross classification design will be illustrated using a two-factor example. The following discussion is similar to that presented by Bennett and Franklin [26]. The EMS are computed from a two-way table in which each row of the table corresponds to a source of variation in the ANOVA table and each column corresponds to a factor. This table for a two-factor crossed design would have four rows and three columns (Table 2). If the factor levels were sampled from populations as described below (factor C denotes nested replication),

Factor	*No. of Levels*	*Population Size*
A	a	α
B	b	β
C	c	γ

then the table and EMS are constructed using the following rules:

(i) In any row write the number of levels under any column which corresponds to a letter not present in the source of variation being considered.
(ii) In any row enter a 1 under any column which corresponds to a letter in parentheses in the source.
(iii) In any row write in the remaining columns

TABLE 2. Expected Mean Square Computations for a Two-Factor Crossed Design with Nested Replication

	FACTORS			EXPECTED MEAN SQUARE COEFFICIENTS (4)			
	A	B	C				
	α*	β	γ	$\sigma^2_{C(AB)}$	σ^2_{AB}	σ^2_{B}	σ^2_{A}
SOURCE	a	b	c				
A	$(1-\frac{a}{\alpha})^{(3)}$	$b^{(1)}$	$c^{(1)}$	$(1-\frac{c}{\gamma})$	$c\,(1-\frac{b}{\beta})$	$-^{(4)}$	bc
B	$a^{(1)}$	$(1-\frac{b}{\beta})^{(3)}$	$c^{(1)}$	$(1-\frac{c}{\gamma})$	$c\,(1-\frac{a}{\alpha})$	ac	-
AB	$(1-\frac{a}{\alpha})^{(3)}$	$(1-\frac{b}{\beta})^{(3)}$	$c^{(1)}$	$(1-\frac{c}{\gamma})$	c	-	-
C(AB)	$1^{(2)}$	$1^{(2)}$	$(1-\frac{c}{\gamma})^{(3)}$	1	-	-	-

* POPULATION SIZE AND NUMBER OF LEVELS, i.e., FACTOR A HAD a LEVELS WHICH WERE SAMPLED FROM A POPULATION OF α POSSIBLE LEVELS.

(1) RESULT OF RULE (i)
(2) RESULT OF RULE (ii)
(3) RESULT OF RULE (iii)
(4) RESULT OF RULES (iv) AND (v), BLANKS RESULT FROM STEP (iv)

$(1 - p/\pi)$ where p and π are the number of levels and population size, respectively, for the factor in the column heading. If $p <<< \pi$, then the coefficient is 1, and if $p = \pi$, then the coefficient is zero.

(iv) The EMS for a given source of variation contains a variance component for each of the other sources which has a letter(s) identical to the letter(s) in the source of interest. For example, the EMS for factor A contains any variance component which has A in its name while the EMS for the AB interaction contains any variance component with the letters AB in its name.

(v) In any EMS the coefficients of the variance components (σ_i^2) will be the product of the entries of the rows in the table ex-excluding those columns which are in the name of the source of variation corresponding to the EMS. Any variance component which has all subscripts occurs with multiplier 1. For example, in computing the EMS for factor A, the A column is excluded ("covered up") while the A and B columns are excluded in computing the coefficients of the σ_i^2's in the EMS for the AB interaction. In the two-factor crossed design, the coefficient for $\sigma^2_{C(AB)}$ is 1 in the EMS for $C(AB)$ because all three columns (A, B, and C) are included in the name of the source.

The EMS's for the two factor crossed design are shown in Table 2. The entries on the left hand side of the table resulted from rules (i), (ii), and (iii), while the coefficients on the right hand side of the table resulted from rules (iv) and (v).

The reader will note that up to this point, finite and infinite populations and fixed and random factors have not been discussed. The EMS algorithms presented by Hicks [28] and Scheffé [30] require that the factors be designated as fixed or random. Bennett and Franklin's [26] algorithm, which has been described here, is more general. Fixed and random factors are the result of assumptions concerning the relationship between the number of levels in the design (p) and the population of levels which could have been sampled (π). If the p levels have been sampled from a finite population of π possible levels and $p = \pi$, then $(1 - p/\pi) = 0$ and the factor is said to be *fixed*. If the p levels represent a sample from an infinitely large population ($\pi = \infty$), then $p/\pi = 0$, $(1 - p/\pi) = 1$, and the factor is *random*. In some cases $p < \pi$ and π is finite. For example, the twenty levels of factor A may represent 20 machines out of a total of 100 possible machines which are manufacturing a given product. In this instance $(1 - p/\pi) = (1 - 20/100) = .8$. The EMS's and F tests for the two factor crossed design in the case of A and B fixed (regression model), A and B random (variance components model), A fixed and B random (random block design or mixed model) are shown in Table 3.

The F test for the significance of a factor effect is MS_F/MS_e, where MS_F is the observed mean square for the factor of interest and MS_e is the error mean square where MS_e is defined as the mean square whose expected value is the same as the expected value of MS_F when $\sigma_F^2 = 0$, i.e., factor F has no significant effect. For example, consider the two factor crossed design with A fixed and B random. From Table 3,

TABLE 3. Expected Mean Squares for Two Factor Crossed Designs*

	A & B FIXED(1)					A & B RANDOM(1)					A FIXED, B RANDOM(1)				
SOURCE	$\sigma^2_{C(AB)}$	σ^2_{AB}	σ^2_{B}	σ^2_{A}	ERROR TERM	$\sigma^2_{C(AB)}$	σ^2_{AB}	σ^2_{B}	σ^2_{A}	ERROR TERM	$\sigma^2_{C(AB)}$	σ^2_{AB}	σ^2_{B}	σ^2_{A}	ERROR TERM
A	1			bc	C(AB)(2)	1	c		bc	AB	1	c		bc	AB
B	1		ac		C(AB)	1	c	ac		AB	1		ac		C(AB)
AB	1	c			C(AB)	1	c			C(AB)	1	c			C(AB)
C(AB)	1					1					1				

*C RANDOM IN ALL CASES, I.E., $(1 - c/\gamma) = 1$; COEFFICIENTS DEVELOPED FROM THE GENERAL FORMULAS SHOWN IN THE RIGHT HAND SIDE OF TABLE 2.

(1)FIXED IMPLIES $(1 - p/\pi) = 0$, RANDOM IMPLIES $(1 - p/\pi) = 1$

(2)THE ERROR MEAN SQUARE FOR TESTING $H_0: \sigma^2_A = 0$ IS $MS_{C(AB)}$
I.E., $E(MS_A) = \sigma^2_{C(AB)} + bc\ \sigma^2_A$

$$E(MS_A) = \sigma^2_{C(AB)} + c\sigma^2_{AB} + bc\sigma^2_A$$

Assuming A has no effect $(H_0 : \sigma^2_A = 0)$

$$E(MS_A \mid \sigma^2_A = 0) = \sigma^2_{C(AB)} + c\sigma^2_{AB}, \quad \text{and}$$

$$E(MS_{AB}) = \sigma^2_{C(AB)} + c\sigma^2_{AB}$$

hence, the significance of the A effect $(H_0 : \sigma^2_A = 0)$ is tested by comparing $F = MS_A/MS_{AB}$ with the tabulated statistic from an F distribution with $(a - 1)$ and $(a - 1)(b - 1)$ degrees of freedom.

From this discussion we can conclude that the EMS's, F ratios, and subsequent inferences from an ANOVA are highly dependent on the assumed relation between number of factor levels in the design (p) and the total number of levels in the population (π) which could have been included in the design. The data in Table 1, which we discussed earlier, are from an experiment in which the effects of 4 reagents and 3 catalysts on production rate (coded) were studied [23]. The F ratios presented by Smith [23] assume both catalysts and reagents are fixed effects and duplicates is a random factor, hence, the error mean square for all effects is the duplicates mean square $(MS_{C(AB)}$, Table 3).

ANOVA TABLE

Source	df	MS		F
Total	23			
Reagents	3	40 ÷ 4	=	10.0
Catalysts	2	24 ÷ 4	=	6.0
R × C	6	14 ÷ 4	=	3.5
Duplicates	12	4		

Since reagents and catalysts were assumed to be fixed factors, we conclude that the reagents and catalyst populations being studied consisted of the four reagents and three catalysts included in this experiment. Any conclusions are restricted to these populations and cannot be generalized to other reagents and catalysts.

Unbalanced Nested Design

The nested design is used frequently to determine the sources of significant variation in a process and to estimate variance components which provide a quantitative measure of the amount of variability contributed by each of the sources. The EMS's for the nested ANOVA in the case of the balanced design can be computed using the Cornfield-Tukey algorithm [3] described earlier. However, many nested designs are unbalanced. The sampling pattern may be unbalanced because of economic constraints or one may be analyzing production data which were collected without the aid of a design. In these situations, the EMS's can be computed using Gaylor and Hartwell's algorithm [9] which, as we noted earlier, gives the same results as the algorithms proposed by Ganguli [7], Anderson and Bancroft [25], and Mahamunulu [17] when all factors are assumed to be random and the populations infinite in size.

The computations for the unbalanced nested design will be illustrated using the data (Table 4) from the "staggered nested design" published by Bainbridge [1]. The experiment was designed to estimate the amount of variability in a chemical property of a textile material which could be attributed to each of four factors.

TABLE 4. Unbalanced Nested Design Data[1]

	MACHINE 1			MACHINE 2
	ANALYST 1		ANALYST 2	ANALYST 3
DAYS	TEST 1	TEST 2	TEST 3	TEST 4
1	6.1	6.6	6.6	8.8
2	8.5	9.6	8.2	8.1
3	8,6	6.7	8.0	7.4
4	9.3	7.2	6.5	8.0
5	8.1	7.1	2.3	9.5
6	8.5	9.0	4.0	9.2
7	9.8	9.8	11.7	12.8
8	9.0	8.0	6.8	9.2
9	11.0	10.9	10.5	11.3
10	9.7	10.6	10.3	9.3
11	10.5	8.4	10.0	4.0
12	8.3	10.6	8.8	9.7
13	8.4	7.2	6.7	4.6
14	10.2	8.0	8.9	2.1
15	9.3	8.7	9.9	9.7
16	7.1	8.7	8.2	10.0
17	5.8	6.8	7.5	10.2
18	8.9	6.6	6.6	9.2
19	11.5	7.1	3.1	10.8
20	10.3	10.0	7.2	9.4
21	9.1	9.5	10.7	10.3
22	5.7	7.7	8.4	10.3
23	8.5	8.8	7.6	8.3
24	9.6	12.2	12.6	11.6
25	9.4	10.4	9.6	9.4
26	10.3	10.6	12.6	11.3
27	7.0	10.6	10.8	11.4
28	11.5	7.3	5.1	9.6
29	6.0	7.0	6.6	2.2
30	8.0	7.0	8.6	6.6
31	13.4	9.2	12.5	11.5
32	12.1	11.7	10.4	9.1
33	14.2	10.6	10.6	4.6
34	10.0	10.4	7.2	7.9
35	6.5	8.4	7.8	9.0
36	6.5	6.8	4.4	8.1
37	9.2	10.1	8.7	9.4
38	11.0	11.0	11.2	10.9
39	8.6	10.0	10.3	9.0
40	8.9	8.0	7.0	7.8
41	6.6	7.2	7.7	9.3
42	8.4	8.8	7.6	6.8

* SEE REFERENCE (1)

	Factors	*Factor Description*
A	Days	Changes in raw material
B in A	Machines	Different production units
C in B	Long Term Test	Different analysts on different shifts using any piece of test equipment
D in C	Short Term Test	Duplicate tests by the same analyst at a given time on one piece of test equipment

Samples were obtained from each of 2 machines selected at random on each of 42 days. The sample from one machine was analyzed by two analysts on different shifts (one analyst in duplicate) while a single determination was made on the sample from the other machine. The ANOVA for the resulting data (Table 4) is given in Table 5.

Before we can construct F tests of significance and and compute the variance components, we have to compute the expected values of the mean squares. The EMS for balanced and unbalanced nested designs have the same form. Namely, the EMS for a given effect contains

(i) σ^2 for the effect itself, plus

(ii) σ^2 for any effect nested in it.

For example, the EMS for factor A in a four factor design contains σ^2's for factors A, B, C, and D (B is nested in A, C in B, and D in C) while the EMS for factor C contains σ^2's for factors C and D (D is nested in factor C) and the EMS for factor D contains only σ_D^2 (no factors are nested in D).

The next step is the computation of the coefficients (k_i's) in the EMS. In this example, the population sizes α, β, γ, and δ are all infinite (random factors), hence, Gaylor and Hartwell's [9] formulae shown at the top of Table 6 reduce to those shown at the bottom of Table 6 which are given by Ganguli [7] and Anderson and Bancroft [25]. If should also be noted that in the case of balanced designs the coefficients computed from these algorithms will be identical to those obtained from Cornfield-Tukey algorithm described earlier. References [8, 12] also discuss EMS computation for unbalanced nested designs.

The coefficients are functions of the sample size tables. Using the dot notation to indicate summation over an index ($n_{i..} = \sum_{jk} n_{ijk}$), the sample size tables for the four factor example are

<table>
<tr><td></td><td colspan="3">1</td><td colspan="3">2</td><td>. . .</td><td colspan="3">42</td></tr>
<tr><td>$n_{i..}$</td><td colspan="3">4</td><td colspan="3">4</td><td></td><td colspan="3">4</td></tr>
<tr><td>$n_{ij.}$</td><td colspan="2">3</td><td>1</td><td colspan="2">3</td><td>1</td><td>. . .</td><td colspan="2">3</td><td>1</td></tr>
<tr><td>n_{ijk}</td><td>2</td><td>1</td><td>1</td><td>2</td><td>1</td><td>1</td><td></td><td>2</td><td>1</td><td>1</td></tr>
</table>

and the resulting coefficients are shown in Table 5. It is interesting to note that the coefficient for σ_C^2, which appears in the EMS for factors A, B, and C, has three different values (3/2, 7/6, 4/3) depending on which EMS is involved (Table 5). The coefficients for σ_B^2 are also different. In a balanced design the coefficient associated with a given σ^2 would be the same in all EMS.

Variance Components

The variance components are computed by equating the observed mean squares to the appropriate EMS's and solving the system of linear equations.

TABLE 5. Analysis of Variance for Unbalanced Nested Design Example

SOURCE	df	SS	MS	EMS
TOTAL	167	751.27		
A DAYS	41	365.58	8.917	$\sigma_D^2 + 3/2\,\sigma_C^2 + 5/2\,\sigma_B^2 + 4\,\sigma_A^2$
B IN A MACHINES	42	196.59	4.681	$\sigma_D^2 + 7/6\,\sigma_C^2 + 3/2\,\sigma_B^2$
C IN B LONG TERM TEST	42	118.79	2.828	$\sigma_D^2 + 4/3\,\sigma_C^2$
D IN C SHORT TERM TEST	42	70.31	1.674	σ_D^2

SOURCE	ERROR df	ERROR MEAN SQUARE	F	VARIANCE COMPONENT	PERCENT
A	26.86(1)	6.300(1)	1.42	.654	14.5
B IN A	49.05(1)	2.684(1)	1.74*	1.331	29.4
C IN B	42.00	1.674	1.69*	.866	19.1
D IN C				1.674	37.0
TOTAL				4.525	

* SIGNIFICANT AT THE .05 PROBABILITY LEVEL

(1) ERROR MEAN SQUARE SYNTHESIZED AND DEGREES OF FREEDOM COMPUTED BY SATTERTHWAITE'S FORMULA (19, 20)

If **M** is the $(p \times 1)$ vector of mean squares, **K** is the $(p \times p)$ matrix of EMS coefficients, and **V** is the $(p \times 1)$ vector of variance components then

$$\mathbf{M} = \mathbf{KV} \text{ implies } \mathbf{V} = \mathbf{K}^{-1}\mathbf{M}, \text{ i.e.}$$

$$\begin{bmatrix} \hat{\sigma}_p^2 \\ \hat{\sigma}_{p-1}^2 \\ \cdot \\ \cdot \\ \cdot \\ \hat{\sigma}_1^2 \end{bmatrix} = \begin{bmatrix} k_{11} & k_{12} & \cdots & k_{1p} \\ k_{21} & k_{22} & \cdots & k_{2p} \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ k_{p1} & k_{p2} & \cdots & k_{pp} \end{bmatrix}^{-1} \begin{bmatrix} MS_1 \\ MS_2 \\ \cdot \\ \cdot \\ \cdot \\ MS_p \end{bmatrix}$$

The variance components (Table 5) in the four factor unbalanced nested design example are:

$$\begin{bmatrix} \hat{\sigma}_D^2 \\ \hat{\sigma}_C^2 \\ \hat{\sigma}_B^2 \\ \hat{\sigma}_A^2 \end{bmatrix} = \begin{bmatrix} 1 & 3/2 & 5/2 & 4 \\ 1 & 7/6 & 3/2 & 0 \\ 1 & 4/3 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}^{-1} \begin{bmatrix} 8.917 \\ 4.681 \\ 2.828 \\ 1.674 \end{bmatrix} = \begin{bmatrix} 1.674 \\ .866 \\ 1.331 \\ .654 \end{bmatrix}$$

The reader will note that the variance components could also be computed by first solving for $\hat{\sigma}_D^2$ and then using that result to solve for $\hat{\sigma}_C^2$, etc. However, it will be shown later that the inverse of the coefficient matrix $(\mathbf{K}^{-1})$ will be useful in making the appropriate F tests. Although it will not be discussed here, variance components in other designs are estimated in the same way as described previously for nested designs, i.e., the observed mean squares are equated to the EMS's and the resulting system of linear equations are solved for the variance components.

Synthesizing Mean Squares

When analyzing nested designs, it is also appropriate to construct F tests to determine whether the observed variance components are significantly different from zero. There is no problem in constructing F tests for balanced nested designs; however, in unbalanced nested designs involving 3 or more factors, there may be one or more effects for which an error mean square does not exist. In these situations, error mean squares can be *synthesized* by forming linear combinations of the observed variance components $(\hat{\sigma}_i^2)$. Since the $\hat{\sigma}_i^2$ are linear combinations of the observed mean squares, the synthesized mean square (L)

$$L = k_1\hat{\sigma}_1^2 + k_2\hat{\sigma}_2^2 + \cdots + k_p\hat{\sigma}_p^2$$

can also be expressed as a linear combination of the observed means squares

$$L = a_1MS_1 + a_2MS_2 + \cdots + a_pMS_p .$$

If ν_i is the degrees of freedom associated with the ith observed mean square (MS_i), then Satterthwaite [19, 20] has shown that the degrees of freedom associated with L are

$$\bar{\nu} = L^2/\sum_{i=1}^{p} (a_iMS_i)^2/\nu_i .$$

TABLE 6. Expected Mean Square Coefficients for Four Factor Unbalanced Nested Designs[1]

		VARIANCE COMPONENT COEFFICIENTS - FINITE POPULATIONS			
SOURCE	df	σ_D^2	σ_C^2	σ_B^2	σ_A^2
A	$a - 1$	1	$\sum_{ijk}\left(n_{ijk}^2 - \frac{n_{ij.}^2}{c_{ij}\gamma}\right)f_i$	$\sum_{ij}\left(n_{ij.}^2 - \frac{n_{i..}^2}{b_i\beta}\right)f_i$	$\Sigma n_{i..}^2 f_i$
B IN A	$\sum_i b_i - a$	1	$\sum_{ijk}\left(n_{ijk}^2 - \frac{n_{ij}}{c_{ij}\gamma}\right)f_{ij}$	$\sum_{ij} n_{ij.}^2 f_{ij}$	
C IN B	$\sum_{ij} c_{ij} - \sum_i b_i$	1	$\sum_{ijk} n_{ijk}^2 f_{ijk}$		
D IN C	$\sum_{ijk} d_{ijk} - \sum_{ij} c_{ij}$	1			

		VARIANCE COMPONENTS COEFFICIENTS - INFINITE POPULATIONS ($\beta = \gamma = \infty$)			
SOURCE	df	σ_D^2	σ_C^2	σ_B^2	σ_A^2
A	$a - 1$	1	$\sum_{ijk} n_{ijk}^2 f_i$	$\sum_{ij} n_{ij.}^2 f_i$	$\sum_i n_{i..}^2 f_i$
B IN A	$\sum_i b_i - a$	1	$\sum_{ijk} n_{ijk}^2 f_{ij}$	$\sum_{ij} n_{ij.}^2 f_{ij}$	
C IN B	$\sum_{ij} c_{ij} - \sum_i b_i$	1	$\sum_{ijk} n_{ijk}^2 f_{ijk}$		
D IN C	$\sum_{ijk} d_{ijk} - \sum_{ij} c_{ij}$	1			

$$f_i = \frac{(1/n_{i..} - 1/n_{...})}{a - 1} \qquad f_{ij} = \frac{(1/n_{ij.} - 1/n_{i..})}{\sum_i b_i - a} \qquad f_{ijk} = \frac{(1/n_{ijk} - 1/n_{ij.})}{\sum_{ij} c_{ij} - \Sigma b_i}$$

[1] a = No. A levels, b_i = No. B levels in i^{th} A class, c_{ij} = No. C levels in ij^{th} B class, d_{ijk} = No. D levels in ijk^{th} C class ($i = 1, 2, \ldots a$; $j = 1, 2, \ldots, b_i$; $k = 1, 2, \ldots, c_{ij}$).

The resulting ratio, $F = MS/L$, is approximately distributed as F with ν and $\bar{\nu}$ degrees of freedom where ν is the degrees of freedom associated with the mean square in the numerator.

The variance components are given by

$$\mathbf{V} = \mathbf{K}^{-1}\mathbf{M}$$

hence, a linear combination of the variance components is given by

$$\mathbf{kV} = \mathbf{kK}^{-1}\mathbf{M}$$

where $\mathbf{k} = (k_1, k_2, \cdots, k_p)$ is the row vector of coefficients in the linear combination of interest. It is concluded that the a_i's for a given synthesized mean square are given by

$$\mathbf{a} = \mathbf{kK}^{-1},$$

$$(a_1\, a_2 \ldots a_p) = (k_1 k_2 \ldots k_p) \begin{bmatrix} k_{11} & k_{12} & \ldots & k_{1p} \\ k_{21} & k_{22} & \ldots & k_{2p} \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ k_{p1} & k_{p2} & \ldots & k_{pp} \end{bmatrix}^{-1}$$

Thus, if we compute the variance components by inverting the variance component coefficient matrix ($\mathbf{K}$), then the $\mathbf{K}^{-1}$ matrix is available and can be used to compute a_i's and subsequently the degrees of freedom ($\bar{\nu}$) associated with the synthesized mean square L. This formulation is particularly useful when one is using a computer to construct the synthesized mean squares and associated approximate F tests [24].

In the four factor unbalanced nested design example (Table 5) the synthesized mean square for testing the H_0: $\sigma_A^2 = 0$ is

$$L = \hat{\sigma}_D^2 + 1.5\,\hat{\sigma}_C^2 + 2.5\,\hat{\sigma}_B^2$$

$$= 1.674 + 1.5(.866) + 2.5(1.331) = 6.300.$$

In terms of mean squares

$$L = MS_D + 1.5(MS_C - MS_D)/4/3$$

$$+ 2.5(MS_B - MS_D - 7/6(MS_C - MS_D/4/3)/3/2$$

$$= 5/3MS_B - 1/3MS_C - 1/3MS_D$$

$$= 5(4.681)/3 - 2.828/3 - 1.674/3 = 6.300$$

hence, the a_i's are: $a_1 = 0$, $a_2 = 5/3$, $a_3 = a_4 = -1/3$.

In matrix notation the a_i's are given by

$$(a_1\ a_2\ a_3\ a_4) = (1\quad 3/2\quad 5/2\quad 0)\begin{bmatrix} 1 & 3/2 & 5/2 & 4 \\ 1 & 7/6 & 3/2 & 0 \\ 1 & 4/3 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}^{-1}$$

$$= (1\quad 3/2\quad 5/2\quad 0)\begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 3/4 & -3/4 \\ 0 & 2/3 & -7/12 & -1/12 \\ 1/4 & -5/12 & 1/12 & 1/12 \end{bmatrix}$$

$$= (0\quad 5/3\quad -1/3\quad -1/3)$$

The degrees of freedom for $L = 6.300$ are

$$\bar{\nu} = \frac{(6.300)^2}{(5(4.681)/3)^2/42 + (2.828/3)^2/42 + (1.674/3)^2/42}$$

$$= 26.86$$

and the ratio $F = 8.917/6.300 = 1.42$ is approximately distributed as F with 41 and 26.86 degrees of freedom. The tabulated F statistic for 40 and 27 degrees of freedom at $\alpha = .10$ is 1.60. It is concluded that $\sigma^2_{\text{Days}} = .654$ is not significantly different from zero. The synthesized mean square and F ratio for factor B (machines) are given in Table 5. The reader is referred to reference [5] for another example of the use of synthesized mean squares in the analysis of unbalanced nested designs.

It is also sometimes necessary to compute synthesized mean squares in the analysis of some balanced designs. The EMS's for a four factor partially nested and crossed design in which factor C is nested in the AB combination and crossed with D are shown in Table 7. If A is a fixed factor and B, C, and D are random factors, then the EMS for factor A

$$E(MS_A) = (1 - c/\gamma)(1 - d/\delta)\sigma^2_{C(AB)D} + (1 - b/\beta)c(1 - d/\delta)\sigma^2_{ABD} + bc(1 - d/\delta)\sigma^2_{AD} + (1 - c/\gamma)d\sigma^2_{C(AB)} + (1 - b/\beta)cd\sigma^2_{AB} + bcd\sigma^2_A$$

reduces to (Table 8)

$$E(MS_A) = \sigma^2_{C(AB)D} + c\sigma^2_{ABD} + bc\sigma^2_{AD} + d\sigma^2_{C(AB)} + cd\sigma^2_{AB} + bcd\sigma^2_A .$$

If one is willing to assume that $\sigma^2_{AD} = 0$, the MS_{AB} can be used as an error term to test H_0: $\sigma^2_A = 0$. In practice, one might test H_0: $\sigma^2_{AD} = 0$ first, by computing $F = MS_{AD}/MS_{ABD}$. If H_0: $\sigma^2_{AD} = 0$ was accepted then H_0: $\sigma^2_A = 0$ could be tested by $F = MS_A/MS_{AB}$. If H_0: $\sigma^2_{AD} = 0$ was rejected, then a mean square and its degrees of freedom would have to be synthesized as discussed earlier. The reader is referred to Scheffé [30, p 245–248] for a discussion of synthesized mean squares in the case of a three way crossed design in which all factors are random.

It should be noted that Satterthwaite's [19, 20] formula for the degrees of freedom of a synthesized

TABLE 7. Expected Mean Squares for a Four Factor Partially Nested and Crossed Design

	FACTOR				EXPECTED MEAN SQUARE COEFFICIENTS (1)								
	$\frac{A}{\alpha^*}$	$\frac{B}{\beta}$	$\frac{C}{\gamma}$	$\frac{D}{\delta}$									
SOURCE	a	b	c	d	$\sigma^2_{C(AB)D}$	σ^2_{ABD}	σ^2_{BD}	σ^2_{AD}	σ^2_D	$\sigma^2_{C(AB)}$	σ^2_{AB}	σ^2_B	σ^2_A
A	$(1-\frac{a}{\alpha})$	b	c	d	$\gamma'\ \delta'$	$\beta' c\delta'$	-	$bc\delta'$	-	$\gamma' d$	β' cd	-	bcd
B	a	$(1-\frac{b}{\beta})$	c	d	$\gamma'\ \delta'$	$\alpha' c\delta'$	$ac\delta'$	-	-	$\gamma' d$	α' cd	acd	
AB	$(1-\frac{a}{\alpha})$	$(1-\frac{b}{\beta})$	c	d	$\gamma'\ \delta'$	$c\ \delta'$	-	-	-	$\gamma' d$	cd		
C (AB)	1	1	$(1-\frac{c}{\gamma})$	d	δ	-	-	-	-	d			
D	a	b	c	$(1-\frac{d}{\delta})$	γ'	$\alpha'\beta' c$	$a\beta' c$	α'bc	abc				
AD	$(1-\frac{a}{\alpha})$	b	c	$(1-\frac{d}{\delta})$	γ'	$\beta' c$	-	bc					
BD	a	$(1-\frac{b}{\beta})$	c	$(1-\frac{d}{\delta})$	γ'	$\alpha' c$	ac						
ABD	$(1-\frac{a}{\alpha})$	$(1-\frac{b}{\beta})$	c	$(1-\frac{d}{\delta})$	γ'	c							
C(AB) D	1	1	$(1-\frac{c}{\gamma})$	$(1-\frac{d}{\delta})$	1								

(1) $\alpha' = (1-\frac{a}{\alpha})$, $\beta' = (1-\frac{b}{\beta})$, $\gamma' = (1-\frac{c}{\gamma})$, $\delta' = (1-\frac{d}{\delta})$

* Population size and number of levels, i.e., Factor A had a levels which were sampled from a population of α possible levels.

TABLE 8. Expected Mean Squares
FOUR FACTOR PARTIALLY NESTED AND CROSSED DESIGN
FACTOR A FIXED, FACTORS B, C, AND D RANDOM

SOURCE	EXPECTED MEAN SQUARE
A	$\sigma^2_{C(AB)D} + c\sigma^2_{ABD} + bc\sigma^2_{AD} + d\sigma^2_{C(AB)} + cd\sigma^2_{AB} + bcd\sigma^2_{A}$
B	$\sigma^2_{C(AB)D} + ac\sigma^2_{BD} + d\sigma^2_{C(AB)} + acd\sigma^2_{B}$
AB	$\sigma^2_{C(AB)D} + c\sigma^2_{ABD} + d\sigma^2_{C(AB)} + cd\sigma^2_{AB}$
C (AB)	$\sigma^2_{C(AB)D} + d\sigma^2_{C(AB)}$
D	$\sigma^2_{C(AB)D} + ac\sigma^2_{BD} + abc\sigma^2_{D}$
AD	$\sigma^2_{C(AB)D} + c\sigma^2_{ABD} + bc\sigma^2_{AD}$
BD	$\sigma^2_{C(AB)D} + ac\sigma^2_{BD}$
ABD	$\sigma^2_{C(AB)D} + c\sigma^2_{ABD}$
C(AB) D	$\sigma^2_{C(AB)D}$

mean square is an approximation and not an exact solution. To this author's knowledge, the accuracy of the approximation has not been investigated in general, although several authors have investigated special cases [2, 4, 10, 15, 16, 31]. It is generally agreed that the approximation is good. If the synthesized mean square is a function of two mean squares, ($L = a_1MS_1 + a_2MS_2$) the approximation is not accurate when there is a large difference between ν_1 and ν_2, the degrees of freedom associated with MS_1 and MS_2 [4], and/or ν_1 and ν_2 are small [14, 15]. Gaylor and Hopper [10] have investigated the accuracy of the approximation when $L = MS_1 - MS_2$.

A final comment is that synthesized mean squares may be negative if one or more of the observed variance components ($\hat{\sigma}_i^2$) is negative. An *ad hoc* procedure is to set any negative $\hat{\sigma}_i^2$ to zero when computing the synthesized mean square. The accuracy of this procedure is unknown to this author.

Computer Programs

Computation of expected mean squares is essential in interpretation of an analysis of variance; however, EMS computations are not available in many ANOVA computer programs. Two programs in the public domain which have EMS options are BMD08V in the UCLA Biomedical Computer Program Package (University of California Press, 2223 Fulton St., Berkeley, California 94720) and the ANOVAR program developed at Brigham Young University.

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References

Periodicals

1. Bainbridge, T. R., "Staggered, Nested Designs for Estimating Variance Components," *Industrial Quality Control*, Vol. 22, No. 1, July 1965, pp. 12–20.
2. Cochran, W. G., "Testing a Linear Relation Among Variances," *Biometrics*, Vol. 7, 1951, pp. 17–32.
3. Cornfield, J., and Tukey, J. W., "Average Values of Mean Squares in Factorials," *Annals of Mathematical Statistics*, Vol. 27, 1956, pp. 907–949.
4. Davenport, J. M., and Webster, J. T., "Type-I Error and Power of a Test Involving a Satterthwaite's Approximate F-Statistic," *Technometrics*, Vol. 14, No. 3, August 1972, pp. 555–569.
5. Eisen, E. J., "The Quasi-F Test for an Unnested Factor in an Unbalanced Hierarchal Design with a Mixed Model," *Biometrics*, Vol. 22, No. 4, December 1966, pp. 937–942.
6. Gaito, J., "Expected Mean Squares in Analysis of Variance Techniques," *Psychological Reports*, Vol. 7, 1960, pp. 3–10.
7. Ganguli, M., "A Note on Nested Sampling," *Sankhyā*, Vol. 5, 1931, pp. 449–452.
8. Gates, C. E., and Shive, C., "The Analysis of Variance of the S-stage Hierarchal Classification," *Biometrics*, Vol. 18, 1962, pp. 529–536.
9. Gaylor, D. W., and Hartwell, T. D., "Expected Mean Squares for Nested Classifications," *Biometrics*, Vol. 25, No. 2, June 1969, pp. 427–430.
10. Gaylor, D. W., and Hopper, F. N., "Estimating the Degrees of Freedom for Linear Combinations of Mean Squares by Satterthwaite's Formula," *Technometrics*, Vol. 11, No. 4, November 1969, pp. 691–706.
11. Gaylor, D. W., Lucas, H. L., and Anderson, R. L., "Calculation of Expected Mean Squares by the Abbreviated Doolittle and Square Root Methods," *Biometrics*, Vol. 26, No. 4, December 1970, pp. 641–656.
12. Gower, J. C., "Variance Component Estimation for Unbalanced Hierarchal Classifications," *Biometrics*, Vol. 18, 1962, pp. 537–542.
13. Hartley, H. O., "Expectation, Variances and Covariances of ANOVA Mean Squares by 'Synthesis,'" *Biometrics*, Vol. 23, No. 1, March 1967, pp. 105–114, 853.
14. Hicks, C. R., "Fundamentals of Analysis of Variance," *Industrial Quality Control*, Vol. 13, 1956, No. 3, pp. 5–8, No. 4, pp. 13–16.
15. Howe, R. B., and Myers, R. H., "An Alternative to Satterthwaite's Test Involving Positive Linear Combinations of Variance Components," *Journal of the*

American Statistical Association, Vol. 65, No. 329, March 1970, pp. 404–412.

16. HUDSON, J. D., AND KRUTCHKOFF, R. G., "A Monte Carlo Investigation of the Size and Power of Tests Employing Satterthwaite's Synthetic Mean Squares," *Biometrika*, Vol. 55, No. 2, 1968, pp. 431–433.
17. MAHAMUNULU, D. M., "Sampling Variances of the Estimates of Variance Components in the Unbalanced 3-Way Nested Classification," Annals of Mathematical Statistics, Vol. 34, 1963, pp. 521–527.
18. RAO, J. N. K., "On Expectations, Variances, and Covariances of ANOVA Mean Squares by 'Synthesis,' " *Biometrics*, Vol. 24, No. 4, December 1968, pp. 963–978.
19. SATTERTHWAITE, F. E., "Synthesis of Variance," *Psychometrika*, Vol. 6, 1941, pp. 309–316.
20. SATTERTHWAITE, F. E., "An Approximate Distribution of Estimates of Variance Components," *Biometrics Bulletin*, Vol. 2, 1946, pp. 110–114.
21. SEARLE, S. R., AND FAWCETT, R. F., "Expected Mean Squares in Variance Components Models having Finite Populations," *Biometrics*, Vol. 26, No. 2, June 1970, pp. 243–254.
22. SCHULTZ, E. F., JR., "Rules of Thumb for Determining Expectations of Mean Squares in Analysis of Variance," *Biometrics*, Vol. 11, No. 2, June 1955, pp. 123–135.
23. SMITH, H., "The Analysis of Data from a Designed Experiment," *Journal of Quality Technology*, Vol. 1, No. 4, October 1969, pp. 259–263.
24. TIETJEN, G. L., AND MOORE, R. H., "On Testing Significance of Components of Variance in the Unbalanced Nested Analysis of Variance," *Biometrics*, Vol. 24, No. 2, June 1968, pp. 423–429.

Books:

25. ANDERSON, R. L., AND BANCROFT, T. A., *Statistical Theory in Research*, McGraw-Hill, New York, 1952, pp. 327–330.
26. BENNETT, C. A., AND FRANKLIN, N. L., *Statistical Analysis in Chemistry and the Chemical Industry*, John Wiley & Sons, Inc., New York, 1954, Chapter 7.
27. BROWNLEE, K., *Statistical Theory and Methodology in Science and Engineering*, First Edition, John Wiley & Sons, New York, 1960, Chapter 13.
28. HICKS, C. R., *Fundamental Concepts in Design of Experiments*, Holt, Rinehart, and Winston, New York, 1964, pp. 153–163.
29. OSTLE, B., *Statistics in Research*, Second Edition, Iowa State University Press, Ames, Iowa, 1963, Chapters 11 and 12.
30. SCHEFFÉ, H., *Analysis of Variance*, John Wiley & Sons, Inc., New York, 1959, pp. 247–248, 284–289.
31. SEARLE, S. R., *Linear Models*, John Wiley & Sons, New York, 1971, pp. 411–413.
32. STEEL, R. D. G., AND TORRIE, J., *Principles and Procedures of Statistics*, McGraw-Hill, New York, 1960, Chapter 11.
33. WINER, B., *Statistical Principles in Experimental Design*, McGraw-Hill, New York, 1962, Chapter 5.